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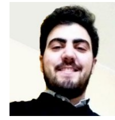

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BTS-Assisted Positioning System - Simulation and Performance Analysis

Abstract
This work concerns the study of a satellite navigation system providing a solution to the positioning problem with the support of a fixed Base Transceiver Station (BTS), which will work as an aiding peer.
The main idea is to use less signals coming from satellites than the minimum required since user's coverage may not be full. The amount of information required to solve the Position, Velocity, Time (PVT) equations will be provided by a fixed aiding peer which provides measurements of its time drift with respect to the GNSS system. The problem was formulated and simulated by using both synthetic satellites positions generated randomly and by using real ones. The results of the study were discussed in terms of accuracy of the position estimation.

Master of Science Academic Project - 02LPXOT *Satellite Navigation Systems* - Professor: **Fabio Dovis**

Contents

1. Introduction

This work concerns the study of a satellite navigation system providing a solution to the positioning problem with the support of a fixed Base Transceiver Station (BTS), which will work as an aiding peer.

One of the main issues with satellite navigation systems discussed most is the availability: the service may not have a good coverage on the user's area.

The main idea is to use less satellites than the minimum amount which is required to localize the user, asking for the help of an aiding peer which provides measurements of its distance from satellites which it has in view.

Such problem was formulated by considering a single user which has exactly three satellites in view, but communicates with a fixed BTS which is close to him, and has a better service coverage, so that it can receive signal from at least another satellite.

The assumption leading to this formulation is that, even if the user position is not exactly the same as the antenna's, it is close enough to let the positioning computation algorithm converge to a solution which is close enough to the true one.

2. Problem Formulation

The problem, by its physical nature, consists of finding the user's coordinates, which, in their relative reference system uniquely identify the user's position.

$$\mathbf{x}_u = [\, x_u \quad y_u \quad z_u \,] \tag{1}$$

The navigation problem is generally solved by computing the distance from each satellite to the user: r_i. Then, by trilateration, the actual user's position is computed.

$$r_i^2 = (x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2 \tag{2}$$

Since distance measurements involve time, there is a fourth unknown: time misalignment between the satellite navigation system time reference and the user's clock: δt_u, which introduces a spatial uncertainty $b_{ut} = c \cdot \delta t_u$.

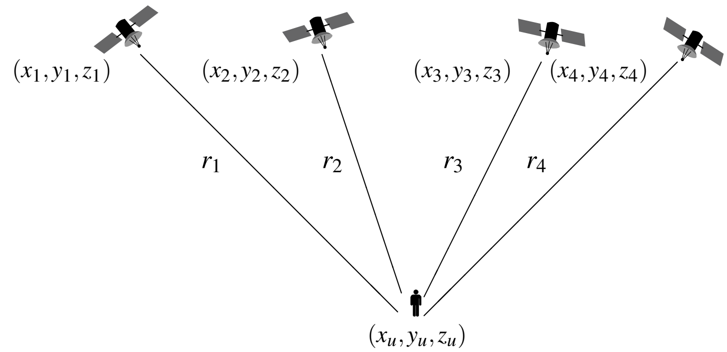

Figure 1. Satellite Positions (x_i, y_i, z_i), User Position (x_u, y_u, z_u) and distances from each satellite r_i.

From now on, distances from satellites will be called pseudoranges ρ_i, since they identify the radius of a sphere centered on each satellite, tangent to the user device.

$$\rho_i = \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2} - b_{ut} \quad (3)$$

The equations leading to the solution of the navigation problem may be linearized through the Taylor expansion around a known location:

$$\hat{\mathbf{x}}_\mathbf{u} = (\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{b}_{ut}) \quad (4)$$

The linearization consists in the following:

$$\Delta\rho_i = \hat{\rho}_i - \rho_i = a_{x_i}\Delta x_u + a_{y_i}\Delta y_u + a_{z_i}\Delta z_u - \Delta b_{ut} \quad (5)$$

$$a_{x_i} = \frac{x_i - \hat{x}_u}{\hat{r}_i}, \qquad a_{y_i} = \frac{y_i - \hat{y}_u}{\hat{r}_i}, \qquad a_{z_i} = \frac{y_i - \hat{y}_u}{\hat{r}_i}$$

$$\hat{r}_i = \sqrt{(x_i - \hat{x}_u)^2 + (y_i - \hat{y}_u)^2 + (z_i - \hat{z}_u)^2}$$

Since the problem consists of finding four unknowns, its solution needs to involve at least four equations, so at least four satellites have to be seen by the user.

$$\begin{cases} \Delta\rho_1 = a_{x_1}\Delta x_u + a_{y_1}\Delta y_u + a_{z_1}\Delta z_u - \Delta b_{ut} \\ \Delta\rho_2 = a_{x_2}\Delta x_u + a_{y_2}\Delta y_u + a_{z_2}\Delta z_u - \Delta b_{ut} \\ \vdots \\ \Delta\rho_n = a_{x_n}\Delta x_u + a_{y_n}\Delta y_u + a_{z_n}\Delta z_u - \Delta b_{ut} \end{cases} \quad (6)$$

In matrix notation, we may write:

$$\Delta\rho = \mathbf{H} \cdot \Delta\mathbf{x_u} \quad (7)$$

By defining the matrices $\Delta\rho, \mathbf{H}, \Delta\mathbf{x_u}$ as follows:

$$\Delta\rho = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \vdots \\ \Delta\rho_n \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} a_{x_1} & a_{y_1} & a_{z_1} & 1 \\ a_{x_2} & a_{y_2} & a_{z_2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{x_n} & a_{y_n} & a_{z_n} & 1 \end{bmatrix}$$

$$\Delta\mathbf{x_u} = \begin{bmatrix} x_u - \hat{x}_u \\ y_u - \hat{y}_u \\ z_u - \hat{z}_u \\ -(b_{ut} - \hat{b}_{ut}) \end{bmatrix} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ -\Delta b_{ut} \end{bmatrix} \quad (8)$$

The solution of the linear equations system can be obtained, by using a Least Mean Square approach, from:

$$\Delta\mathbf{x_u} = (\mathbf{H}^T \cdot \mathbf{H})^{-1}\mathbf{H}^T\Delta\rho \quad (9)$$

The position obtain in this way may be used to compute iteratively the actual user's position with higher accuracy, by setting

$$\hat{\mathbf{x}}_\mathbf{u} = (\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, \hat{b}_{ut} - \Delta b_{ut}) \quad (10)$$

Pseudorange measurements are affected by noise, assumed to be a realization of a random variable:

$$\delta\rho_i \sim \mathcal{N}(0, \sigma^2_{UERE}), \text{ i.i.d.} \quad (11)$$

The standard deviation of the measurements is called σ_{UERE}: User Equivalent Range Error.

Since the pseudorange measurements are affected by noise, at the end of the computation, the position estimation will be the realization of random variable affected by bias and variance.

$$\Delta\rho + \delta\rho = \mathbf{H} \cdot (\Delta\mathbf{x_u} + \delta\mathbf{x_u}) \quad (12)$$

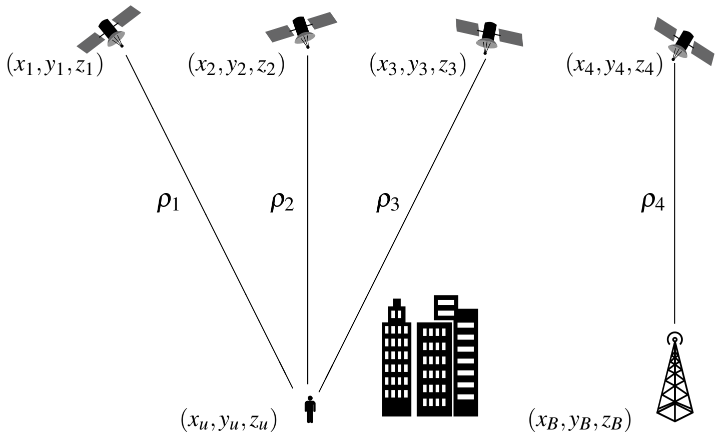

Figure 2. Satellite Positions (x_i, y_i, z_i), User Position (x_u, y_u, z_u), BST Position (x_B, y_B, z_B) and relative pseudoranges ρ_i.

The average magnitude of the error committed in considering the computed one as true position, is called bias of the estimation.

$$\begin{aligned} \textbf{Bias}\ [\hat{\mathbf{x}}_\mathbf{u}] &= \mathbb{E}[\hat{\mathbf{x}}_\mathbf{u} - \mathbf{x}_\mathbf{u}] = \mathbb{E}[\delta \mathbf{x}_\mathbf{u}] \\ &= (\mathbf{H}^T \cdot \mathbf{H})^{-1} \mathbf{H}^T \cdot \mathbb{E}[\delta \rho] = 0 \end{aligned} \tag{13}$$

While the covariance of the estimated position is:

$$\begin{aligned} \textbf{COV}\ [\delta \mathbf{x}_\mathbf{u}] &= (\mathbf{H}^T \cdot \mathbf{H})^{-1} \mathbf{H}^T \cdot \textbf{COV}[\delta \rho] \cdot \\ &\quad \mathbf{H}(\mathbf{H}^T \cdot \mathbf{H})^{-1} \\ &= (\mathbf{H}^T \cdot \mathbf{H})^{-1} \cdot \sigma^2_{UERE} \end{aligned} \tag{14}$$

One of the most interesting quantity of interest is the trace matrix $\mathbf{G} = (\mathbf{H}^T \cdot \mathbf{H})^{-1}$, because it determines the Geometric Diluition of Precision (GDOP):

$$\textbf{GDOP} = \sqrt{\mathrm{tr}\{(\mathbf{H}^T \cdot \mathbf{H})^{-1}\}} \tag{15}$$

The variance of the error will be defined by using this coefficient, which refers to the geometry of the satellites with respect to the user:

$$\begin{aligned} \sigma_{\hat{\mathbf{x}}_u} &= \sqrt{\sigma^2_{x_u} + \sigma^2_{y_u} + \sigma^2_{z_u} + \sigma^2_{x_{b_{ut}}}} \\ &= \textbf{GDOP} \cdot \sigma_{UERE} \end{aligned} \tag{16}$$

Of course it is desirable that the variance of the estimation is not too high, so that the computed position converges to the right one.

Once considered the the problem formulation, more restrictive assumptions may be done.

By supposing that user has three satellites in view, it has to communicate with an aiding peer which has a fourth satellite in view, both measuring pseudoranges from the satellites they see.

Once the base station has computed its own pseudorange, transmits it to the user, who employs it to find its own position.

Given that the user position is not the same as the base station's, the fourth pseudorange ρ_4 will introduce errors because it refers to:

$$\mathbf{x_B} = (x_B, y_B, z_B, b_{bt}) \tag{17}$$

The model built in this project tries to give a solution to the localization problem, by using as first approximation point the base station location:

$$(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{b}_{ut}) = (x_B, y_B, z_B, b_{bt}) \tag{18}$$

Given that if the user and the antenna are not supposed to be synchronous, it is important to take into account the ratio:

$$\gamma_{ub} = \frac{\Delta b_{bt}}{\Delta b_{ut}}. \tag{20}$$

By considering that the antenna may have a lower misalignment with respect to the GNSS time reference, we may set $\gamma_{ub} = 1$ as a worst condition.

The linear system of equations may be written, in matrix notation, as:

$$\Delta \mathbf{D} \rho = \mathbf{DH} \cdot \Delta \mathbf{x_u} \tag{21}$$

$$\begin{cases} \Delta\rho_1 - \Delta\rho_2 = (a_{x_1} - a_{x_2})\Delta x_u + (a_{y_1} - a_{y_2})\Delta y_u + (a_{z_1} - a_{z_2})\Delta z_u \\ \Delta\rho_1 - \Delta\rho_3 = (a_{x_1} - a_{x_3})\Delta x_u + (a_{y_1} - a_{y_3})\Delta y_u + (a_{z_1} - a_{z_3})\Delta z_u \\ \Delta\rho_1 - \Delta\rho_4 = (a_{x_1} - a_{x_4})\Delta x_u + (a_{y_1} - a_{y_4})\Delta y_u + (a_{z_1} - a_{z_4})\Delta z_u - (\Delta b_{ut} - \Delta b_{bt}) \\ \Delta\rho_1 = a_{x_1}\Delta x_u + a_{y_1}\Delta y_u + a_{z_1}\Delta z_u - \Delta b_{ut} \end{cases} \quad (19)$$

By defining respectively the matrices $\mathbf{\Delta D}\rho$ and $\mathbf{DH}$:

$$\mathbf{\Delta D}\rho = \begin{bmatrix} \Delta\rho_1 - \Delta\rho_2 \\ \Delta\rho_1 - \Delta\rho_3 \\ \Delta\rho_1 - \Delta\rho_4 \\ \Delta\rho_1 \end{bmatrix} \quad (22)$$

$$\mathbf{DH} = \begin{bmatrix} a_{x_1} - a_{x_2} & a_{y_1} - a_{y_2} & a_{z_1} - a_{z_2} & 0 \\ a_{x_1} - a_{x_3} & a_{y_1} - a_{y_3} & a_{z_1} - a_{z_3} & 0 \\ a_{x_1} - a_{x_4} & a_{y_1} - a_{y_4} & a_{z_1} - a_{z_4} & 1 - \gamma_{ub} \\ a_{x_1} & a_{y_1} & a_{z_1} & 1 \end{bmatrix}$$

This time, the solution to the positioning problem is given by:

$$\mathbf{\Delta x_u} = (\mathbf{DH})^{-1} \cdot \mathbf{\Delta D}\rho \quad (23)$$

Since the matrix $\mathbf{DH}$ is square, it results to be invertible provided that it is non-singular.

3. Performance Evaluation

In order to evaluate the performances of such an estimator, the error on pseudorange measurements has to be taken into account:

$$\mathbf{\Delta D}\rho + \delta\mathbf{D}\rho = \mathbf{DH} \cdot (\mathbf{\Delta x_u} + \delta\mathbf{x_u}) \quad (24)$$

First of all, from the physical model:

$$\delta\mathbf{D}\rho = \begin{bmatrix} \delta D\rho_1 \\ \delta D\rho_2 \\ \delta D\rho_3 \\ \delta D\rho_4 \end{bmatrix} = \begin{bmatrix} \delta\rho_1 - \delta\rho_2 \\ \delta\rho_1 - \delta\rho_3 \\ \delta\rho_1 - \delta\rho_{4_{(B)}} \\ \delta\rho_1 \end{bmatrix} \quad (25)$$

It is possible to notice that now the components of $\delta\mathbf{D}\rho$ are not i.i.d.:

$$\begin{cases} \delta D\rho_i \sim \mathcal{N}(0, 2\sigma^2_{UERE}), \quad i = 1,2 \\ \delta D\rho_3 \sim \mathcal{N}(\delta\rho_{4,geom}, 2\sigma^2_{UERE}) \\ \delta D\rho_4 \sim \mathcal{N}(0, \sigma^2_{UERE}) \end{cases} \quad (26)$$

In this formulation, the pseudorange ρ_4, being measured by the antenna and not by the user, carries two error components:

$$\delta\rho_{4_{(B)}} = \delta\rho_{4_{(U)}} + \delta\rho_{4,geom} \quad (27)$$

It is important to notice that the two errors have different nature:

$$\begin{cases} \delta\rho_{4_{(U)}} \sim \mathcal{N}(0, \sigma^2_{UERE}) \\ \delta\rho_{4,geom}, \text{ Deterministic} \\ \delta\rho_{4_{(B)}} \sim \mathcal{N}(\delta\rho_{4,geom}, \sigma^2_{UERE}) \end{cases} \quad (28)$$

In order to have a metric of the error, Bias and Variance of the error has to be computed.

$$\begin{aligned} \mathbf{Bias}\,[\hat{\mathbf{x}}_\mathbf{u}] &= \mathbb{E}[\hat{\mathbf{x}}_\mathbf{u} - \mathbf{x_u}] = \mathbb{E}[\delta\mathbf{x_u}] \\ &= (\mathbf{DH})^{-1} \cdot \mathbb{E}[\delta\mathbf{D}\rho] \\ &= (\mathbf{DH})^{-1} \cdot [\,0,\ 0,\ \delta\rho_{4,geom},\ 0\,]^T \end{aligned} \quad (29)$$

$$\mathbf{COV}[\delta\mathbf{x_u}] = (\mathbf{DH})^{-1}\mathbf{COV}[\delta\mathbf{D}\rho](\mathbf{DH}^T)^{-1} \quad (30)$$

The steps to the actual computation of the covariance matrix are reported in Equations 31, 32.

The previous can be simplified in the following expression:

$$\mathbf{COV}[\delta\mathbf{D}\rho] = \mathbf{C_1} \cdot \sigma^2_{UERE} + \mathbf{C_2} \cdot \delta\rho^2_{4,geom} \quad (33)$$

where $\mathbf{C_1}$ and $\mathbf{C_2}$ are opportunely set to:

$$\mathbf{C_1} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (34)$$

$$\mathbf{C_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (35)$$

The covariance of the error can be expressed easier:

$$\mathbf{COV}[\delta\mathbf{x_u}] = (\mathbf{DH})^{-1} \cdot \mathbf{COV}[\delta\mathbf{D}\rho] \cdot (\mathbf{DH}^T)^{-1} \quad (36)$$

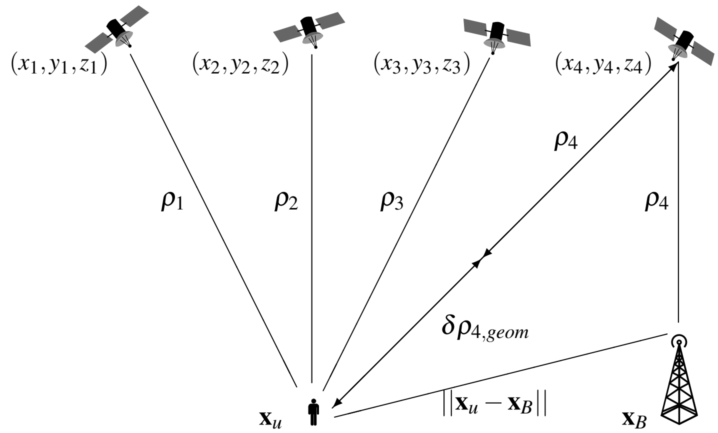

Figure 3. Satellite Positions (x_i, y_i, z_i), User Position $\mathbf{x}_u$, BST Position $\mathbf{x}_B$ and relative pseudoranges ρ_i.

$$\mathbf{COV}[\delta\mathbf{D}\rho] = \begin{bmatrix} \mathbb{E}[\delta D\rho_1^2] & \mathbb{E}[\delta D\rho_1 \delta D\rho_2] & \mathbb{E}[\delta D\rho_1 \delta D\rho_3] & \mathbb{E}[\delta D\rho_1 \delta D\rho_4] \\ \mathbb{E}[\delta D\rho_2 \delta D\rho_1] & \mathbb{E}[\delta D\rho_2^2] & \mathbb{E}[\delta D\rho_2 \delta D\rho_3] & \mathbb{E}[\delta D\rho_2 \delta D\rho_4] \\ \mathbb{E}[\delta D\rho_3 \delta D\rho_1] & \mathbb{E}[\delta D\rho_3 \delta D\rho_2] & \mathbb{E}[\delta D\rho_3^2] & \mathbb{E}[\delta D\rho_3 \delta D\rho_4] \\ \mathbb{E}[\delta D\rho_4 \delta D\rho_1] & \mathbb{E}[\delta D\rho_4 \delta D\rho_2] & \mathbb{E}[\delta D\rho_4 \delta D\rho_3] & \mathbb{E}[\delta D\rho_4^2] \end{bmatrix} \tag{31}$$

$$\mathbf{COV}[\delta\mathbf{D}\rho] = \begin{bmatrix} \mathbb{E}[\delta\rho_1^2] + \mathbb{E}[\delta\rho_2^2] & \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] \\ \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] + \mathbb{E}[\delta\rho_3^2] & \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] \\ \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] + \mathbb{E}[\delta\rho_{4_{(B)}}^2] & \mathbb{E}[\delta\rho_1^2] \\ \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] & \mathbb{E}[\delta\rho_1^2] \end{bmatrix} \tag{32}$$

From here, we may define a new quantity called **DGDOP** which takes into account the Diluition on Precision that we achieve by using this technique.

$$\mathbf{DG} = (\mathbf{DH})^{-1} \cdot (\mathbf{C_1}) \cdot (\mathbf{DH}^T)^{-1} \tag{37}$$

$$\mathbf{GG} = (\mathbf{DH})^{-1} \cdot (\mathbf{C_2}) \cdot (\mathbf{DH}^T)^{-1} \tag{38}$$

The variance of the error will be defined by using the following quantities:

- **DGDOP** which takes into account: the geometry of the satellites with respect to the user, the ratio γ_{ub} due to the user and BTS time misalignment and the matrix **C** which comes out from our problem formulation;

- **GGDOP** which takes into account the geometric error due to the use of a pseudorange which is relative to a satellite seen by the aiding peer.

In fact, the overall standard deviation of the estimation $\hat{\mathbf{x}}_u$ can be expressed as the euclidean norm of the standard deviations along its coordinates:

$$\begin{aligned} \sigma_{\hat{\mathbf{x}}_u} &= \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + \sigma_{x_{b_ut}}^2} \\ &= \sqrt{\mathbf{DGDOP}^2 \cdot \sigma_{UERE}^2 + \mathbf{GGDOP}^2 \cdot \delta\rho_{4,geom}^2} \end{aligned} \tag{39}$$

that take into account the traces of the two matrices **DG** and **GG**:

$$\mathbf{DGDOP} = \sqrt{\mathrm{tr}(\mathbf{DG})}, \qquad \mathbf{GGDOP} = \sqrt{\mathrm{tr}(\mathbf{GG})} \tag{40}$$

By doing this, the two effects previously described are taken into account.

4. Simulation

The simulation of the model discussed in the previous sections was made in Matlab®,

4.1 Synthetic Data Simulation

In the first instance of simulations, it is simulated a scenario where:

- The Aiding Peer is located in the origin of the reference system: $\mathbf{x}_B = [0, 0, 0, \hat{b}_{bt}]$.

- The User is located at a fixed distance from the antenna. $\mathbf{x}_u = \mathbf{x}_B + \left[500, 500, 0, \frac{\hat{b}_{bt}}{\gamma_{ub}}\right]$

- The Satellite Vehicles (SV) have random position: $\mathbf{x}_{SV} \sim \mathcal{N}(0, \sqrt{2e7})$

- The SV follow a random linear trajectory. (The speed of each SV has been altered from its realistic value, in order to see what happens with a smaller/larger variation in the skyplot).

- The solution of the problem is obtained by using an algorithm which solves the Linear System of equations iterating on pseudorange measurements:

  1. Measure pseudoranges from the satellites in view;
  2. (a) If they are at least 4, solve Equation (9).
     (b) If they are 3 instead, get the pseudorange measurement from the BTS and solve Equation (23).
  3. (a) If the position estimation obtained at step (2) has small enough bias, ($||\hat{\mathbf{x}}_u - \mathbf{x}_u|| < 10$[m]) return the computed position.
     (b) If the bias is not much large, update the approximation point to the computed one and go to step (1).
     (c) If the computed position is way too far from the BTS, ($||\hat{\mathbf{x}}_u - \mathbf{x}_B|| > 1500$[m]), it means that the algorithm does not converge to a position which would be covered by the BTS.

- The bias of the computed position is shown in magnitude an on the single coordinates.

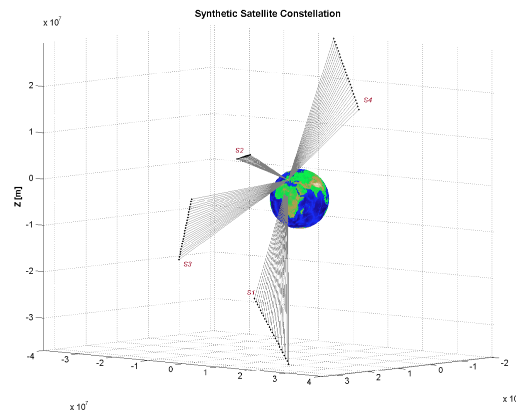

Figure 4. Random constellation of Satellites

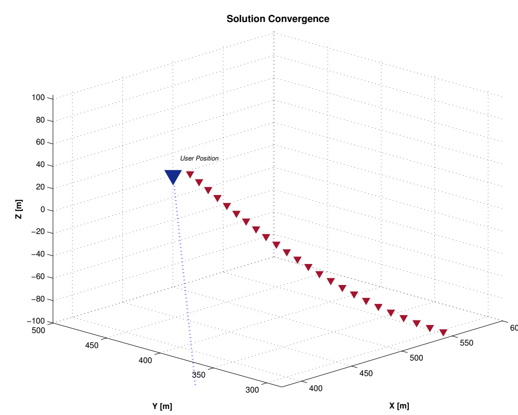

Figure 5. Convergence of the solution

As shown in Figure 4, by using a random generated constellation of satellites following a random linear trajectory, it is possible see in Figure 5 that the computation of user's position converges even if it lasts long.

Since the solution will be affected by bias, on all the coordinates, in Figure 6 it is have reported the overall bias that the user experiences:

$$\mathbb{E}[||\mathbf{x_u} - \hat{\mathbf{x}}_\mathbf{u}||] = \sqrt{(x_u - \hat{x}_u)^2 + y_u - \hat{y}_u)^2 + (z_u - \hat{z}_u)^2} \tag{41}$$

Then, in Figure 7 it is reported the bias on the single coordinates:

$$\mathbb{E}[\mathbf{x_u} - \hat{\mathbf{x}}_\mathbf{u}] = [(x_u - \hat{x}_u),\ (y_u - \hat{y}_u),\ (z_u - \hat{z}_u)]. \tag{42}$$

As previously shown, for synthetic SV positions, the algorithm that iteratively solves PVT equations actually converges.

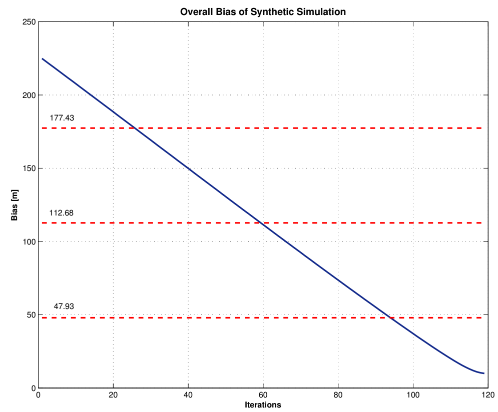

Figure 6. Bias of the norm of the solution.

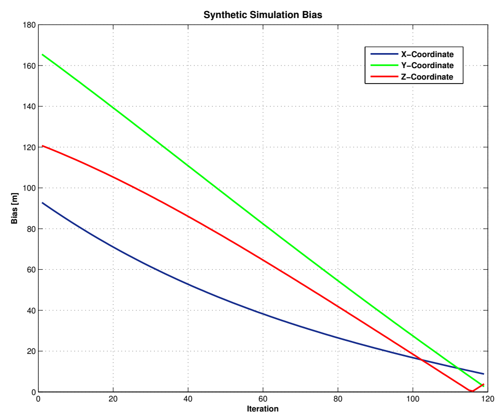

Figure 7. Bias on the coordinates of the solution.

4.2 Real Data Simulation

In order to give more consistency to the simulation, it is made with real data, by the following steps:

- Measurements are obtained by a GPS receiver located here at Polytechnic of Turin, ECEF coordinates are rougly $\mathbf{x_u} = [4.47,\ 0.60,\ 4.49,\ 0.73] \cdot 10^6$ [m], since the measurements obtained were discontinuous, a subset of the whole data collection (where they have a continuous behaviour) was selected.
- One of the pseudorange measurements is altered as if it was obtained by an aiding peer located in $\mathbf{x}_B = \mathbf{x}_U \cdot [1,1,1,\gamma_{ub}] + [500,500,0,0]$.
- The solution of the problem is obtained by using the same algorithm as previously.
- The bias of the computed position is shown in magnitude and on ECEF coordinates.

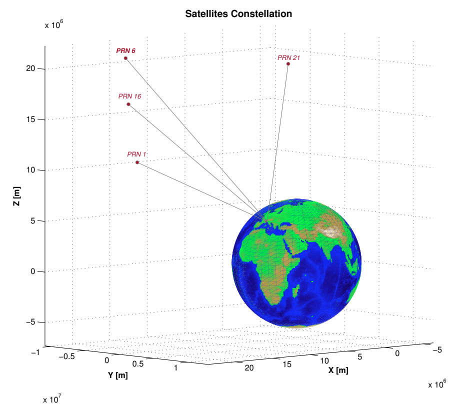

Figure 8. Real satellites constellation

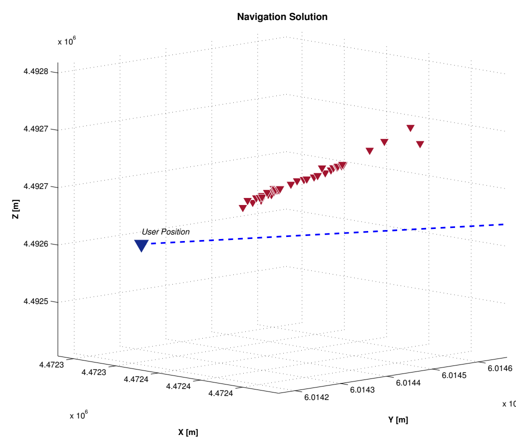

Figure 9. Real position estimation

In Figure 8 are reported the real satellites and user's position. As it is possible to appreciate in Figure 9, the positioning solution gets very close to the actual position, but is always biased.

In Figure 10 is reported the overall bias of the simulation, whereas in Figure 11 is reported the single-coordinate bias.

As we can immediately see from the plots, the scenario generated with random SV positions gives a sharper bias curve.

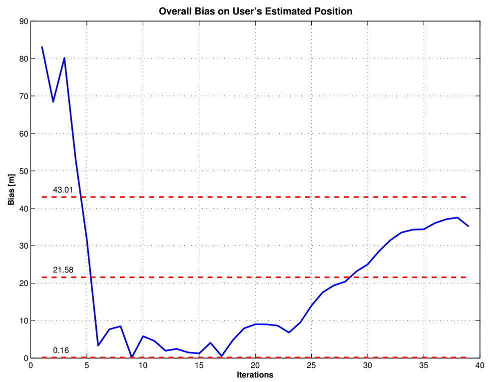

Figure 10. Bias of the norm of the solution

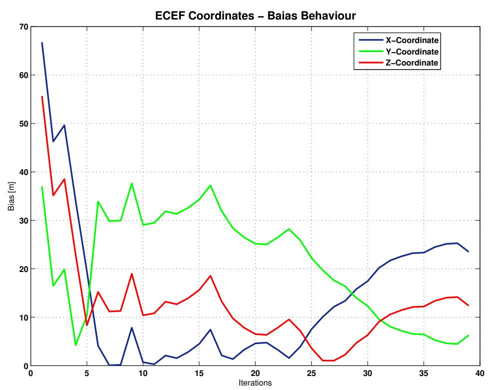

Figure 11. Bias of the norm of the solution

5. Summary of Results

The algorithm was tested by running a Monte Carlo simulation, with the generation of random satellites' positions following random trajectories.

The output of the simulation declared about 60% convergency of the implemented solution.

Once the performances of such a solution are obtained by simulating both with synthetic and real data, the aim is to study when the new parameters introduced achieve better performances or exploit most the utility the solution.

5.1 Time Drift of the two peers

First of all, consider the issues involved with the parameter $\gamma_{ub} = \frac{\Delta b_{bt}}{\Delta b_{ut}}$.

In order to do that, fix the Satellite constellation standing above the user and the aiding peer. Focused on an optimal geometry, reported in Figure 12: the user has in view three equispaced satellites at the horizon, while the fourth is seen by the antenna.

From the known results (29), let the user to obtain the fourth measure from a satellite close to the zenith of the aiding peer because it introduces less bias.

Let γ_{ub} vary in [0, 2], in order to understand what happens when the misalignment ratio is different.

One could expect that having a very small value of γ_{ub} would lead to better performances, since it introduces less error in pseudorange measurement.

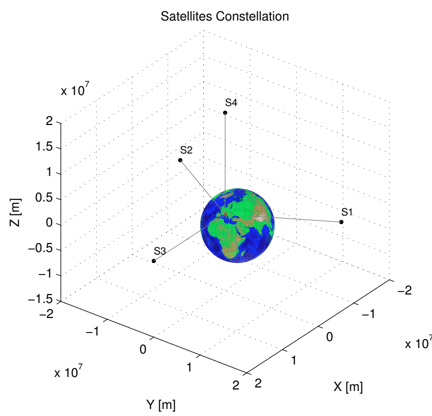

Figure 12. Optimal Constellation of Satellites

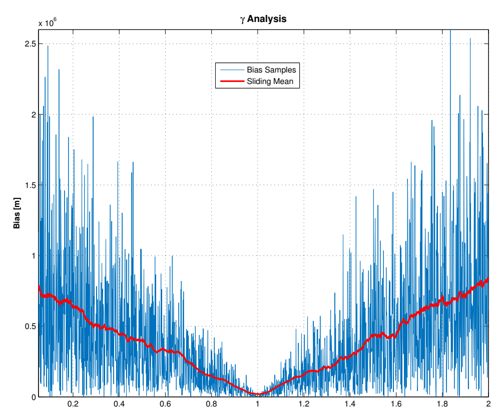

Figure 13. Bias$[\hat{\mathbf{x}}_\mathbf{u}]$ varying with $\gamma_{ub} \in [0,2]$.

Executing the same algorithm for different values of γ_{ub}, it is obtained the plot reported in Figure 13, where it is possible to see that a very low value of bias is obtained for $\gamma_{ub} = 1$.

The reason of this is not so immediate by looking at formulas, but results to be pretty easy to expect one thinks that the algorithm solves the LS of equations to find a value b_{ut}.

If the pseudorange ρ_4 measured by the BTS is affected by exactly the same time misalignment as the other ones measured by the user, the matrix $\mathbf{DH}$ has the same coefficients on all the rows, this makes the search for a b_{ut} value much easier.

So, having an aiding receiver with time drift which is way too far from the user's leads to a worse estimation of the position.

5.2 Distance of the two peers

Another important issue of our problem is how much the bias and the variance vary when the distance of the user from the aiding peer vary.

The algorithm was run with $||\mathbf{x}_B - \mathbf{x_u}|| \in [0, 1500]$ meters, by considering the coverage of a standard telecommunication BTS.

From the output shown in Figure14, it is possible to notice that, the theoretical value assumed by the bias depends mostly on the User-BTS distance on the slant range ($\delta\rho_{4,geom}$).

In particular, it is going to assume higher values when the measurements from the SV used as aid to the user is located close to the horizon.

Satellites at the zenith of the aiding peer not visible by the user would carry a lower bias. This may find application in urban environments.

After these considerations, it is possible to discuss applicability of such a solution.

The main idea is to understand how much it is consistent for a user to get a position even if it is affected by a lot of bias. It may be useful to introduce a new metric for the Optimality of our solution:

$$\psi_{U,B} = \frac{\mathbb{E}[||\hat{\mathbf{x}}_\mathbf{u} - \mathbf{x_u}||]}{||\mathbf{x}_B - \mathbf{x_u}||} \tag{43}$$

that is a ratio between amount of bias we have at a given distance by the aiding peer.

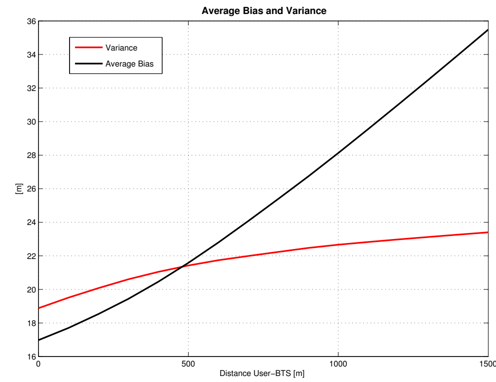

Figure 14. Bias and Variance or the position estimation varying with User-BTS distance

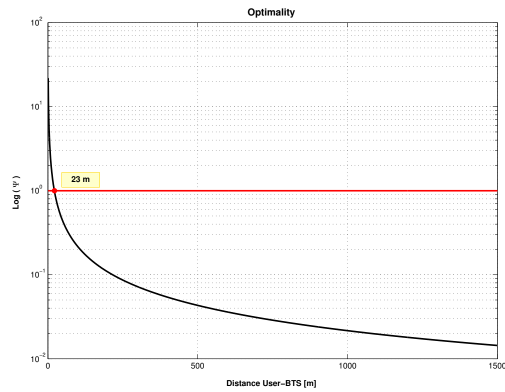

Figure 15. Trade-off of the applicability of our solution over distance

This consideration leads to a trade-off between the mean error that the user is disposed to accept with respect to the possibility of getting positioned where the aiding peer is.

In this simulation, it is supposed that when the error is larger than the distance from the next given point, it is convenient for him to get the position of that point.

With this assumption, it is possible to found out that when the factor $\psi_{U,B} < 1$, the user is so far from the aiding peer that the bias he gets by implementing this solution is less than the one he would accept by being located at the BTS position.

From real data simulations, as shown in Figure 15, $\psi_{U,B} = 1$ at about 23 meters, so it is convenient for users who are further than this distance.

6. Further comments

The implementation of such a solution may involve further considerations, which exploit the benefits of using of a fixed aiding peer.

- At first, the BTS has a known position. By using just one satellite in view, it may compute its own time misalignment with respect to the GNSS time reference.

  This may be used to adjust its clock and keep the value assumed by Δb_{bt} much smaller than the one assumed by Δb_{ut}.

  Moreover, the antenna may use a receiver whose clock frequency drifts less from the reference one, with respect to the user's.

  This leads to the consideration that the time misalignment ratio $\gamma_{ub} \to 0$, but this is way far from the ideal condition ($\gamma_{ub} = 1$), where the bias assumes lower values.

  A hint to solve this may be the compensation of the value Δb_{bt} so that it is close to the one assumed by Δb_{ut}, for example by applying this aiding technique in a synchronous network infrastructure.

- The user may receive more than a single pseudorange measurement from the aiding peer, and use them with an LMS approach to extend our solution and solve the positioning problem.

  This may be of very feasible since the aiding peer may be located where there is a good GNSS coverage and have a wider view of the skyplot.

- The antenna may help the user to improve the solution of the linear system of equations (or to select the best subset of pseudoranges) by transmitting the measurement (or the measurements) from the SV(s) it has in view close to the zenith.

  This may be useful because they introduce less error, given that the User-BTS distance in slant range distance is smaller.

- Obtaining a measured approximation of $\delta\rho_{4,geom}$ may help to compensate the error.

  This lead to have a zero-mean third component in the bias vector, so that the user can appreciate a higher accuracy in the position.

- Using more than one aiding peer may be an interesting perspective since it would help very much even if it is difficult to have different sets of SV in view in a small region on the Earth.

  Moreover, the aiding peers have to communicate so that the user receives different sets of measures from different sets of SVs.

As far as we can see, getting extra information from an aiding peer may help to get coverage where the service is not available, but involves issues about time misalignment and geometry, so the user should be satisfied as much as the error may be compensated.